14-3 The Vector (Cross) Product

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If $\vec{u} = <u_1, u_2, u_3>$ and $\vec{v} = <v_1, v_2, v_3>$,

then the vector or cross product is written $\vec{u}\times\vec{v}$ and is defined as:

$$\vec{u}\times\vec{v} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} i - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} j + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} k$$

A cross product of 2 vectors is another vector (
scalar) which is perpendicular to both the origi
vectors.

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Ex1. $\vec{a} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ $\vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\vec{c} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$ Find:

a.) $\vec{b}\times\vec{c}$

$$\begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 2 & 0 & 4 \end{vmatrix} = i\begin{vmatrix} 2 & 3 \\ 0 & 4 \end{vmatrix} - j\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} + k\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix}$$

$$= i(8-0) - j(4-6) + k(0-4)$$

$$= 8i - j(-2) - 4k$$

$$= 8i + 2j - 4k \quad \boxed{\begin{pmatrix} 8 \\ 2 \\ -4 \end{pmatrix}}$$

b.) $\vec{a}\cdot(\vec{b}\times\vec{c})$

$$\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 8 \\ 2 \\ -4 \end{pmatrix} = 16 + 2 + 4 = 22$$

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Properties of Cross Products

1.) If $\vec{u}$ and $\vec{v}$ are parallel, then $\vec{u}\times\vec{v} = 0$

2.) $\vec{u}\times\vec{v} = -(\vec{v}\times\vec{u})$

3.) $\vec{u}\times(\vec{v}\pm\vec{w}) = \vec{u}\times\vec{v} \pm \vec{u}\times\vec{w}$

4.) $i\times j = k$ and $i\times k = j$ and $j\times k = i$

5.) $\vec{u}\cdot(\vec{u}\times\vec{v}) = 0$ (note: $\vec{u}\times\vec{v}$ is $\perp$ to $\vec{u}$)

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Definition of Cross Product

$\vec{u}\times\vec{v}$ is a vector perpendicular to both $\vec{u}$ and $\vec{v}$ obeying the right-hand rule shown and has the magnitude:

$$|\vec{u}\times\vec{v}| = |\vec{u}|\cdot|\vec{v}|\cdot\sin\theta$$

Hence, the angle between 2 vectors is given by:

$$\sin\theta = \frac{|\vec{u}\times\vec{v}|}{|\vec{u}|\cdot|\vec{v}|}$$

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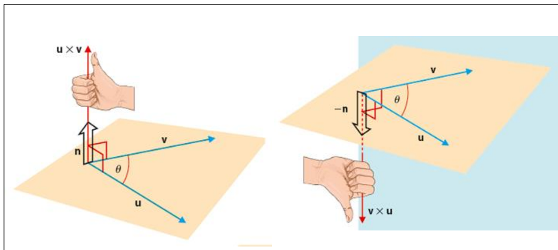

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Ex1. $\vec{a}=\begin{pmatrix}2\\3\\-1\end{pmatrix}$ $\vec{b}=\begin{pmatrix}-1\\2\\1\end{pmatrix}$ Find:

a.) A unit vector that is perpendicular to both $\vec{a}$ and $\vec{b}$

$\vec{a}\times\vec{b}\ \begin{vmatrix} i & j & k\\ 2 & 3 & -1\\ -1 & 2 & 1\end{vmatrix} = i\begin{vmatrix}3 & -1\\2 & 1\end{vmatrix} - j\begin{vmatrix}2 & -1\\-1 & 1\end{vmatrix} + k\begin{vmatrix}2 & 3\\-1 & 2\end{vmatrix}$

$i(3+2)-j(2-1)+k(4+3)$

$5i - 1j + 7k$

$\sqrt{25+1+49}$
$=\sqrt{75}$

$< \frac{5}{\sqrt{75}}, \frac{-1}{\sqrt{75}}, \frac{7}{\sqrt{75}} >$

$<\frac{5}{5\sqrt{3}}, \frac{-1}{5\sqrt{3}}, \frac{7}{5\sqrt{3}}> = <\frac{1}{\sqrt{3}}, \frac{-1}{5\sqrt{3}}, \frac{7}{5\sqrt{3}}>$

b.) The measure of the angle between vectors $\vec{a}$ and $\vec{b}$

$\sin^{-1}\left(\frac{\sqrt{75}}{\sqrt{14}\sqrt{6}}\right) = 70.9°$

c.) Show that $\vec{a}\times\vec{b}$ is $\perp$ $\vec{a}$ and $\vec{a}\times\vec{b}$ is $\perp$ $\vec{b}$

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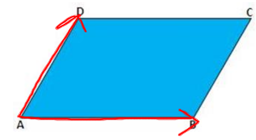
Cross Products in Areas

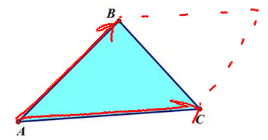
Parallelogram ABCD

$Area = |\overrightarrow{AB}\times\overrightarrow{AD}|$



Triangle ABC

$Area = \frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$



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Ex4. Calculate the area of the triangle with vertices A (3,-5,1), B(-1,1,3), and C(-1,5,2).

$\overrightarrow{AB} = <-4,6,2>$ $\overrightarrow{BC} = <0,4,-1>$

$\overrightarrow{AB}\times\overrightarrow{BC} = \begin{vmatrix} i & j & k\\ -4 & 6 & 2\\ 0 & 4 & -1\end{vmatrix} = i\begin{vmatrix}6 & 2\\4 & -1\end{vmatrix} - j\begin{vmatrix}-4 & 2\\0 & -1\end{vmatrix} + k\begin{vmatrix}-4 & 6\\0 & 4\end{vmatrix}$

$= i(-6-8) - j(4-0) + k(-16-0)$

$= -14i - 4j - 16k$

$|\overrightarrow{AB}\times\overrightarrow{BC}| = \sqrt{(-14)^2+(-4)^2+(-16)^2} = \sqrt{196+16+256}$

$= \sqrt{468} = 6\sqrt{13}$

$Area = \frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{BC}| = \frac{1}{2}(6\sqrt{13}) = 3\sqrt{13}$ units^2

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Ex5. The points A(3,1,2), B(-1,1,5), and C(7,2,3) are vertices of parallelogram ABCD.

a.) Find the coordinates of D

$\overrightarrow{AB} = <-4,0,3>$
$\overrightarrow{BC} = <8,1,-2>$

$D = A + \overrightarrow{BC}$
$D = (3,1,2) + <8,1,-2>$
$\boxed{D = (11,2,0)}$

b.) Calculate the area of the parallelogram.

$\overrightarrow{BA}\times\overrightarrow{BC} = \begin{vmatrix} i & j & k\\ -4 & 0 & 3\\ -8 & -1 & 2\end{vmatrix} = i(0+3) - j(-8+24) + k(4\ldots$

$3i - 16j + 4k$

$|\overrightarrow{BA}\times\overrightarrow{BC}| = \sqrt{9+(-16)^2+16} = 16.76\ u^2$

$\sqrt{281}$

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The Scalar Triple Product

$\vec{a}\cdot(\vec{b}\times\vec{c})$

$\vec{a}=\begin{pmatrix}2\\1\\-1\end{pmatrix}$ $\vec{b}=\begin{pmatrix}1\\2\\3\end{pmatrix}$ $\vec{c}=\begin{pmatrix}2\\0\\4\end{pmatrix}$

In Example 1, $\vec{a}\cdot(\vec{b}\times\vec{c}) = 22$

Alternate calculation,

$$\begin{vmatrix}2 & 1 & -1\\1 & 2 & 3\\2 & 0 & 4\end{vmatrix} = 2\begin{vmatrix}2 & 3\\0 & 4\end{vmatrix} - 1\begin{vmatrix}1 & 3\\2 & 4\end{vmatrix} + -1\begin{vmatrix}1 & 2\\2 & 0\end{vmatrix}$$

$= 2(8-0) - 1(4-6) - 1(-4)$

$= 16 + 2 + 4 = 22$

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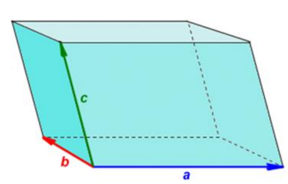
Triple Scalar Product in Volumes

Parallelpiped



$Volume = |\vec{a}\cdot(\vec{b}\times\vec{c})|$

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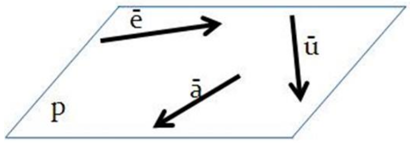
Hence, if three vectors are coplanar, the triple scalar product is 0 (volume of 0).



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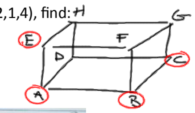
Ex6. Parallelpiped ABCDEFGH has vertices A(1,0,2), B (2,3,3), C(-3,-1,2), and E(2,1,4), find: H

a.) The coordinates of D, F, G, & H.



$\vec{AB} = \langle 1, 3, 1 \rangle$
$F = E + \vec{AB} = (1+2) + (3+1) + (1+4)$
$F = (3, 4, 5)$
$D = A + \vec{BC} = (-5+1) + (0+-4) + (2+-1)$
$D = (-4, -4, 1)$
$G = F + \vec{BC} = (3+-5) + (4-4) + (5-1)$
$G = (-2, 0, 4)$
$H = E + \vec{BC} = (2+-5) + (1+-4) + (4+-1)$
$H = (-3, -3, 3)$

b.) The volume of ABCDEFGH.

$$= \begin{vmatrix} i & j & k \\ -5 & -4 & -1 \\ 1 & 3 & 1 \end{vmatrix} = \begin{vmatrix} -4 & -1 \\ 3 & 1 \end{vmatrix} i - \begin{vmatrix} -5 & -1 \\ 1 & 1 \end{vmatrix} j + \begin{vmatrix} -5 & -4 \\ 1 & 3 \end{vmatrix} k$$
$$(-4+-3)i - (-5++1)j + (-15++4)k$$
$$(1, 1, 2) \cdot (-1i + 4j - 11k)$$
$$|-1 + 4 - 22| = |-19| = 19$$

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Ex7. Show that the points A(1,2,3), B(-1,5,6), C(3,7,5), and D(-3,-8,-1) are coplanar.

$\vec{AB} = \begin{pmatrix} -2 \\ 3 \\ 3 \end{pmatrix}$ $\vec{AC} = \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}$ $\vec{AD} = \begin{pmatrix} -4 \\ -10 \\ -4 \end{pmatrix}$

$$\vec{AB} \cdot (\vec{AC} \times \vec{AD}) = \begin{vmatrix} -2 & 3 & 3 \\ 2 & 5 & 2 \\ -4 & -10 & -4 \end{vmatrix}$$

$$= -2\begin{vmatrix} 5 & 2 \\ -10 & -4 \end{vmatrix} - 3\begin{vmatrix} 2 & 2 \\ -4 & -4 \end{vmatrix} + 3\begin{vmatrix} 2 & 5 \\ -4 & -10 \end{vmatrix}$$

$$= -2(-20+20) - 3(-8+8) + 3(-20+20)$$
$$= -2(0) - 3(0) + 3(0) = 0$$

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Torque

Torque, moment or moment of force is the tendency of a force to rotate an object about an axis, fulcrum, or pivot. Just as a force is a push or a pull on an object, a torque can be thought of as a twist to an object. Mathematically, torque is defined as the cross product of the lever-arm distance and force, which tends to produce rotation. Torque is a measure of the turning force on an object such as a bolt or a flywheel. For example, pushing or pulling the handle of a wrench connected to a nut or bolt produces a torque (turning force) that loosens or tightens the nut or bolt.

$$\tau = r \times F$$
$$\tau = |r| \cdot |F| \cdot \sin\theta$$

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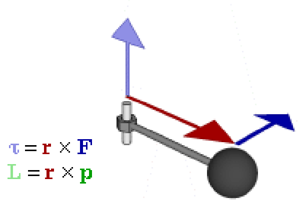

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